

Kadomtsev-Petviashvili equation: Nonlinear self-adjointness and conservation laws

Nail H. Ibragimov

Department of Mathematics and Science, Blekinge Institute of Technology,
371 79 Karlskrona, Sweden

Abstract. The method of nonlinear self-adjointness is applied to the Kadomtsev-Petviashvili equation. The infinite set of conservation laws associated with the infinite algebra of Lie point symmetry of the KP equation is constructed.

Keywords: KP equation, nonlinear self-adjointness, Lie point symmetries, Conservation laws.

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1 Lie point symmetries of the KP equation

It is convenient for our purposes to write the Kadomtsev-Petviashvili [1] equation

$$D_x(u_t - uu_x - u_{xxx}) = u_{yy}, \quad (1.1)$$

or

$$u_{tx} - uu_{xx} - u_x^2 - u_{xxxx} = u_{yy},$$

in the form of the system (see [2] and the references therein)

$$u_t - uu_x - u_{xxx} - \omega_y = 0, \quad \omega_x - u_y = 0. \quad (1.2)$$

The Lie algebra of the infinitesimal symmetries of the KP equation (1.1) as well as of the system (1.2) is quite similar to the infinite-dimensional symmetry Lie algebra of the Lin-Reissner-Tsien equation [3]

$$2\varphi_{tx} + \varphi_x \varphi_{xx} = \varphi_{yy}, \quad (1.3)$$

describing the non-steady state gas flow with transonic velocities. The symmetry Lie algebra of Eq. (1.3) has been obtained in [4] (see also [5], §28). It involves five arbitrary functions of t and contains, in particular the following operators:

$$\begin{aligned} X_f = & 3f(t) \frac{\partial}{\partial t} + (f'(t)x + f''(t)y^2) \frac{\partial}{\partial x} + 2f'(t)y \frac{\partial}{\partial y} \\ & + [f''(t)x^2 + 2f'''(t)xy^2 + \frac{1}{3}f^{(4)}(t)y^4 - f'(t)\varphi] \frac{\partial}{\partial \varphi}, \end{aligned} \quad (1.4)$$

$$X_g = g(t) \frac{\partial}{\partial y} + g'(t)y \frac{\partial}{\partial x} + [2g''(t)xy + \frac{2}{3}g'''(t)y^3] \frac{\partial}{\partial \varphi}, \quad (1.5)$$

$$X_h = h(t) \frac{\partial}{\partial x} + [2h'(t)x + 2h''(t)y^2] \frac{\partial}{\partial \varphi}. \quad (1.6)$$

The system (1.2) admits the following operators (compare with (1.4)-(1.6)):

$$\begin{aligned} X_f = & 3f \frac{\partial}{\partial t} + (f'x + \frac{1}{2}f''y^2) \frac{\partial}{\partial x} + 2f'y \frac{\partial}{\partial y} \\ & - [2f'u + f''x + \frac{1}{2}f'''y^2] \frac{\partial}{\partial u} - [3f'\omega + f''yu + f'''xy + \frac{1}{6}f^{(4)}y^3] \frac{\partial}{\partial \omega}, \end{aligned} \quad (1.7)$$

$$X_g = 2g \frac{\partial}{\partial y} + g'y \frac{\partial}{\partial x} - g''y \frac{\partial}{\partial u} - [g'u + g''x + \frac{1}{2}g'''y^2] \frac{\partial}{\partial \omega}, \quad (1.8)$$

$$X_h = h \frac{\partial}{\partial x} - h' \frac{\partial}{\partial u} - h''y \frac{\partial}{\partial \omega}, \quad (1.9)$$

where f, g, h are three arbitrary functions of t . We will ignore the obvious symmetry

$$X_\alpha = \alpha(t) \frac{\partial}{\partial \omega}$$

of the system (1.2) describing the addition to ω an arbitrary function of t .

Note, that the operators (1.7)-(1.9) considered without the term $\frac{\partial}{\partial \omega}$ span the infinite-dimensional Lie algebra of symmetries of the KP equation (1.1). They coincide (up to normalizing coefficients) with the symmetries of the KP equation that were first obtained by F. Schwarz in 1982 (see also [6] and [7]).

2 Nonlinear self-adjointness

The Kadomtsev-Petviashvili equation written in the form (1.1) or in the form of the system (1.2) does not have a Lagrangian. Let us investigate the KP equation for nonlinear self-adjointness [8].

The formal Lagrangian for the system (1.2) is written

$$\mathcal{L} = v(u_t - uu_x - u_{xxx} - \omega_y) + z(\omega_x - u_y). \quad (2.1)$$

The reckoning shows that

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta u} &= -v_t + uv_x + v_{xxx} + z_y, \\ \frac{\delta \mathcal{L}}{\delta \omega} &= v_y - z_x. \end{aligned}$$

Hence we can write the adjoint system to (1.2) in the form

$$v_t - uv_x - v_{xxx} - z_y = 0, \quad z_x - v_y = 0. \quad (2.2)$$

Eqs. (2.2) become identical with the KP equations (1.2) upon the substitution

$$v = u, \quad z = \omega. \quad (2.3)$$

It means that the system (1.2) is nonlinearly self-adjoint, specifically it is quasi self-adjoint ([8], Section 1.6, Definition 1.3).

3 Conservation laws provided by Lie point symmetries

Let us introduce the notation

$$x^1 = t, \quad x^2 = x, \quad x^3 = y, \quad u^1 = u, \quad u^2 = \omega$$

and write conservation laws in the form of the differential equation

$$[D_t(C^1) + D_x(C^2) + D_y(C^3)]_{(1.2)} = 0, \quad (3.1)$$

where $|_{(1.2)}$ means that the equation holds on the solutions of the system (1.2).

We will use the general formula given in [8] for constructing the conserved vector associated with symmetry

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}$$

of a system of differential equations with a classical or formal Lagrangian \mathcal{L} . Since the maximum order of derivatives involve in formal Lagrangian \mathcal{L} given by Eq. (2.1) is equal to three, this formula is written

$$\begin{aligned} C^i = & \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) \right] \\ & + D_j (W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) \right] + D_j D_k (W^\alpha) \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}, \end{aligned}$$

where

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha.$$

We will apply the above formula to the symmetries (1.7)-(1.9). Invoking that the system (1.2) is nonlinearly self-adjoint with the substitution (2.3), we will replace in C^i the *non-physical variables* v and z with u and ω , respectively, thus arriving to conserved vectors for the KP system. Since the formal Lagrangian (2.1) vanishes on the solutions of the system (1.2), we can omit the term $\xi^i \mathcal{L}$ and take the formula for the conserved vector in the following form:

$$\begin{aligned} C^i = & W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) \right] \\ & + D_j (W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) \right] + D_j D_k (W^\alpha) \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}, \end{aligned} \quad (3.2)$$

where

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad \alpha = 1, 2. \quad (3.3)$$

Using in (3.2) the expression (2.1) for \mathcal{L} and eliminating v and z by means of Eqs. (2.3) we obtain

$$\begin{aligned} C^1 &= u W^1, \\ C^2 &= -(u^2 + u_{xx}) W^1 + \omega W^2 + u_x D_x(W^1) - u D_x^2(W^1), \\ C^3 &= -\omega W^1 - u W^2. \end{aligned} \quad (3.4)$$

The expressions (3.3) for the operator (1.7) are written:

$$W^1 = -3fu_t - (2u + xu_x + 2yu_y)f' - \left(x + \frac{1}{2}y^2u_x\right)f'' - \frac{1}{2}y^2f''', \quad (3.5)$$

$$W^2 = -3f\omega_t - (3\omega + x\omega_x + 2y\omega_y)f' - \left(yu + \frac{1}{2}y^2\omega_x\right)f'' - xyf''' - \frac{1}{6}y^3f^{(4)}.$$

Substituting W^1 given by (3.5) in the first equation (3.4) and eliminating u_t by using the first equation (1.2) we obtain

$$\begin{aligned} C^1 = & -3(u^2u_x + uu_{xxx} + u\omega_y)f - (2u^2 + xuu_x + 2yu_y)f' \\ & - \left(xu + \frac{1}{2}y^2uu_x\right)f'' - \frac{1}{2}y^2uf'''. \end{aligned} \quad (3.6)$$

Now we single out the total derivatives with respect to x and y , by taking into account the second equation (1.2), and rewrite (3.6) in the form

$$C^1 = -\frac{1}{2}f'u^2 - \left(xf'' + \frac{1}{2}y^2f'''\right)u + D_x(P) + D_y(Q), \quad (3.7)$$

where

$$\begin{aligned} P = & \left[\frac{3}{2}u_x^2 + \frac{3}{2}\omega^2 - u^3 - 3uu_{xx}\right]f - \frac{1}{2}f'xu^2 - \frac{1}{4}f''y^2u^2, \\ Q = & -3fu\omega - f'yu^2. \end{aligned} \quad (3.8)$$

Thus, the first component of the conserved vector can be reduced to the form

$$\tilde{C}^1 = -\frac{1}{2}f'u^2 - \left(xf'' + \frac{1}{2}y^2f'''\right)u. \quad (3.9)$$

Remark 3.1. In reducing (3.6) to the form (3.7) we use simple identities such as

$$\begin{aligned} uu_{xxx} &= D_x(uu_{xx}) - u_xu_{xx} = D_x\left(uu_{xx} - \frac{1}{2}u_x^2\right), \\ u\omega_y &= D_y(u\omega) - \omega u_y = D_y(u\omega) - \omega\omega_x = D_y(u\omega) - D_x\left(\frac{1}{2}\omega^2\right), \\ 2yu_y &= D_y(yu^2) - u^2. \end{aligned}$$

To find the second component of the conserved vector, we substitute the expressions (3.5) of W^1, W^2 in the second equation (3.4), add $D_t(P)$ with P

defined in (3.8) and obtain:

$$\begin{aligned}\bar{C}^2 &= C^2 + D_t(P) \\ &= (uu_{xx} + \frac{1}{3}u^3 - \frac{1}{2}u_x^2 - \frac{1}{2}\omega^2)f' + (xu_{xx} + \frac{1}{2}xu^2 - u_x)f'' \\ &\quad + \frac{1}{4}(y^2u^2 + 2y^2u_{xx} - 4xy\omega)f''' - \frac{1}{6}y^3\omega f^{(4)} + D_y(R),\end{aligned}$$

where

$$R = (2yuu_{xx} + \frac{2}{3}yu^3 - yu_x^2 - y\omega^2 - xu\omega)f' - \frac{1}{2}y^2u\omega f''. \quad (3.10)$$

Thus, the second component of the conserved can be reduced to

$$\begin{aligned}\tilde{C}^2 &= (uu_{xx} + \frac{1}{3}u^3 - \frac{1}{2}u_x^2 - \frac{1}{2}\omega^2)f' + (xu_{xx} + \frac{1}{2}xu^2 - u_x)f'' \\ &\quad + \frac{1}{4}(y^2u^2 + 2y^2u_{xx} - 4xy\omega)f''' - \frac{1}{6}y^3\omega f^{(4)}.\end{aligned} \quad (3.11)$$

Finally, the third component of the conserved vector is obtained by substituting the expressions (3.5) of W^1, W^2 in the third equation (3.4) and adding $D_t(Q), D_x(R)$ with Q, R defined in (3.8), (3.8):

$$\tilde{C}^3 = C^3 + D_t(Q) + D_x(R).$$

The reckoning yields:

$$\tilde{C}^3 = u\omega f' + x\omega f'' + (xyu + \frac{1}{2}y^2\omega)f''' + \frac{1}{6}y^3u f^{(4)}. \quad (3.12)$$

Ignoring the tilde in the quantities (3.9), (3.11), (3.12), we summarize the result in the following statement.

Proposition 3.1. The infinitesimal symmetry (1.7) of the Kadomtsev-Petviashvili equations (1.1) provides the conserved vector $C = (C^1, C^2, C^3)$ with the components

$$\begin{aligned}C^1 &= -\frac{1}{2}f'u^2 - (xf'' + \frac{1}{2}y^2f''')u, \\ C^2 &= (uu_{xx} + \frac{1}{3}u^3 - \frac{1}{2}u_x^2 - \frac{1}{2}\omega^2)f' + (xu_{xx} + \frac{1}{2}xu^2 - u_x)f'' \\ &\quad + \frac{1}{4}(y^2u^2 + 2y^2u_{xx} - 4xy\omega)f''' - \frac{1}{6}y^3\omega f^{(4)}, \\ C^3 &= u\omega f' + x\omega f'' + (xyu + \frac{1}{2}y^2\omega)f''' + \frac{1}{6}y^3u f^{(4)}.\end{aligned} \quad (3.13)$$

Remark 3.2. The validity of the conservation equation (3.1) for the vector (3.13) follows from the following equation:

$$\begin{aligned}
& D_t(C^1) + D_x(C^2) + D_y(C^3) \\
&= (uf' + xf'' + \frac{1}{2}y^2f''')(u_{xxx} + uu_x + \omega_y - u_t) \\
&+ (\omega f' + xyf''' + \frac{1}{6}y^3f^{(4)})(u_y - \omega_x).
\end{aligned} \tag{3.14}$$

The similar calculations for the operators (1.8) and (1.9) yield the following.

Proposition 3.2. The symmetry (1.8) of the system (1.1) provides the conserved vector $C = (C^1, C^2, C^3)$ with the components

$$\begin{aligned}
C^1 &= yug'', \\
C^2 &= \left(x\omega - yu_{xx} - \frac{1}{2}yu^2\right)g'' + \frac{1}{2}y^2\omega g''', \\
C^3 &= -(xu + y\omega)g'' - \frac{1}{2}y^2ug'''.
\end{aligned} \tag{3.15}$$

Proposition 3.3. The symmetry (1.9) of the system (1.1) provides the conserved vector $C = (C^1, C^2, C^3)$ with the components

$$\begin{aligned}
C^1 &= uh', \\
C^2 &= y\omega h'' - \left(u_{xx} + \frac{1}{2}u^2\right)h', \\
C^3 &= -\omega h' - yuh''.
\end{aligned} \tag{3.16}$$

Different approaches to construction of conservation laws for the KP equation can be found, e.g. in [9, 10, 11]. In particular, the Lagrangian approach and the Noether theorem are used in the paper [11] which contains an interesting discussion of the infinite set of conservation laws. Note that the second equation of the system (1.2) guarantees that the vector field (u, ω) has the potential ϕ defined by $u = \phi_x$, $\omega = \phi_y$. Then the system (1.2) is replaced by the *potential KP equation*

$$\phi_{xt} - \phi_x\phi_{xx} - \phi_{xxxx} - \phi_{yy} = 0 \tag{3.17}$$

which, unlike equation (1.1) or the system (1.2), has a Lagrangian, namely

$$L = -\frac{1}{2}\phi_x\phi_t + \frac{1}{6}\phi_x^3 + \frac{1}{2}\phi_y^2 - \frac{1}{2}\phi_{xx}^2. \tag{3.18}$$

Now the Noether theorem can be used upon rewriting the symmetries of the KP equation in terms of the potential ϕ . This approach is used in the paper [12] which contains profound results on the conservation laws associated with the infinite algebra of Lie point symmetries of Eq. (3.17). In particular, it is demonstrated that the differential and integral forms of the conservation laws are equivalent only when the functions $f(t), g(t), h(t)$ in the symmetries (1.7)-(1.9) are low-order polynomials. For the details I refer the reader to [12].

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